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Report 4

EFFECT OF A STEP LOAD MOVING WITH CONSTANT SUPERSEISMIC VELOCITY ON A HALF-SPACE OF A VARIABLE MODULUS MATERIAL

by

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FOREWORD

The current work on variable moduli models is part of Contract No. DACA39-67-C-0048, "Investigation of Ground Shock Effects in Nonlinear Hysteretic Media", being conducted for the U.S. Army Engineer Waterways Experiment Station (WES) under DASA sponsorship.

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ABSTRACT

The report considers the two dimensional effects of a step wave progressing with constant superseismic velocity on the surface of a half-space. The material treated is isotropic and it is assumed that incremental relations between both deviatoric and volumetric stress and strain depend not only on the instantaneous values of those quantities, but also on bulk and shear moduli which differ according to whether initial loading, unloading, or reloading occurs.

Simple closed form solutions are obtained when the moduli K and G are constant and lie within certain limits. For the more general case, when K and G are functions of the first and second invariants of stress, solutions requiring only quadratures are found.

As an auxiliary study, a problem involving a half-space of fluid with a bilinear pressure volume relation is solved.

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LIST OF SYMBOLS*)

c_p , c_S	Velocity of propagation of elastic P-waves and S-waves.
e_{ij}	Strain deviators.
G, K	Shear and bulk moduli, respectively.
G_o , G_1 , K_o	Constants used in expressions for bulk and shear moduli.
J_1 , J_2	Invariants, Eqs. (5) and (6).
p	Nondimensional pressure.
s_x , s_y , s_1 , s_2 , s_{ij}	Stress deviators with respect to axes x, y, etc.
t	Time.
u, v	Nondimensional velocities.
V	Velocity of surface load.
v_x , v_y	Velocities with respect to axes x and y.
X	Nondimensional quantity defined by Eq. (17).
x, y, z, ξ, η, ζ	Coordinates defined in the text.
γ	Shear strain.
Δσ, Δv, etc.	Increments of σ, v, etc. at a discontinuity.
ϵ_{kk}	Volumetric strain.
θ, δ	Angles related to directions of principal stress defined in the text.
ρ	Density.
σ_x , σ_y , σ_1 , σ_2 , σ_{ij}	Stress components with respect to axes x, y, etc.
τ	Shear stress.
ϕ, ϕ_P , ϕ_S , $\bar{\phi}_S$	Position angles measured from the surface and defined in the text.

*) Other symbols in the text are defined as they occur.

I INTRODUCTION.

In References [1] and [2] a material model was described which can be fitted reasonably well to the behavior of soils in uniaxial strain and triaxial shear tests. The model was introduced for the investigation of wave propagation and is characterized by the fact that the stress-strain diagrams for typical loading-unloading cycles show hysteresis loops. This is achieved by prescribing different bulk and shear moduli in loading and unloading. The model is thus an alternative to elastic-plastic ones. The new model does not require any statement equivalent to a yield condition. The following will only give details required in the present investigation. For full information the reader is referred to the above references.

In order to gain an understanding concerning the propagation of dynamic disturbances a previous study, Ref. [3], presented closed form solutions for one dimensional propagation of plane waves. The present study is concerned with the plane problem of the two dimensional effects of a step wave progressing with supersonic velocity on the surface of a half-space. Solutions of such two dimensional problems by other than purely numerical, finite difference methods not only add to the understanding of the problem, but permit also a check on the correctness and effectiveness of numerical codes for the multidimensional situations. The desire for such a check was the prime motivation for the present investigation.

The material treated is isotropic and the model assumes that the incremental relations between volumetric stress and strain, and between deviatoric stress and strain depend not only on the instantaneous values of these quantities, but also on the question of whether the change in these quantities occurs during initial loading, unloading, or reloading. The stress-strain relations appear thus in form exactly as in a conventional elastic material

$$\dot{\varepsilon}_{kk} = \frac{1}{3K} \dot{J}_1 \tag{1}$$

and

$$\dot{e}_{ij} = \frac{1}{2G} \dot{s}_{ij} \qquad i \neq j \tag{2}$$

where J_1 , ε_{kk} represent the first invariants of stress and strain, respectively, and s_{ij} , e_{ij} are the deviatoric stress and strain components. The moduli are expressed in terms of the invariants of the stress tensor. K is a function of J_1 , while G may be a function of J_1 and J_2

$$K = K_o + k(J_1) \tag{3}$$

$$G = G_o + g(J_2 , J_1) \tag{4}$$

where

$$J_1 = \sigma_1 + \sigma_2 + \sigma_3 \tag{5}$$

$$J_2 = \frac{1}{2} s_{ij} s_{ij} \tag{6}$$

However, the dependency on J_1 and J_2 is expressed by different functions for initial loading, unloading, or reloading.

Consider the relation between J_1 and ε_{kk} where the value 3K represents the slope of the J_1 - ε_{kk} diagram. For a material which initially hardens such a diagram will be in general of the form of Fig. (1a). The path 1-2 represents initial loading, while 2-3 represents unloading*), where energy considerations require that point 3 be to the right of the initial path 1-2. On reloading to point 2, according to the model, the material will retrace the path 3-2 until point 2 is reached. On further loading, say point 4, it follows the initial loading law. The simplest case occurs when the functions K are constants, different for loading and unloading. The resulting ε_{kk} - J_1 diagram is shown in Fig. (1b).

Consider, for simplicity the situation when J_1 as well as the deviators s_{ij} in a Cartesian coordinate system ξ,η,ζ are kept constant when i=j. Only one of the shears $s_{\xi\eta} \equiv \tau$ is varied, while the other two, $s_{\xi\zeta} = s_{\eta\zeta}$, vanish. To study the relation between τ and the corresponding strain deviator $e_{\xi\eta} \equiv \gamma$, requires simultaneous consideration of the diagrams of J_2 and τ, Figs. (2a,b). Initial loading is represented by 1-2. On unloading, 2-3, the invariant J_2 will reach a minimum value for $\tau_3 = 0$. Further unloading in τ represents reloading, as J_2 increases again, until at point 4 the value J_2 equals the previous maximum J_2. Further decrease in τ, path 4-5, brings

*) Because of the intended application to soils, the model implies that J_1 remains compressive.

the initial loading law for G into force, as shown in Figs. (2a,b). If the functions G are constants, the τ-γ diagram consists of straight lines, as shown in Fig. (2c).

It will be subsequently shown that simple, nearly trivial closed form solutions for the problem to be studied can be obtained if the functions K and G are constants, and lie between certain limits. For the more general case where K and G are functions of J_1 and J_2 solutions requiring only quadratures will be found.

Prior to studying a solid half-space of the new model material under progressing loads, it seemed educational to consider a similar model for fluids, with pressure volume relations according to Fig. (1). For progressing step loads only uninteresting results are obtained, because in these cases the pressure field becomes simply a leading shock front (where the pressure jumps) followed by a uniform pressure. The result is thus entirely unaffected by the unloading law. The latter would affect the results, however, if the pressure decays, say linearly, after the initial sudden rise. Results in closed form for bilinear fluids were obtained and are presented in Appendix B. They may be a useful guide in further studies concerning solids subjected to decaying loads.

II FORMULATION OF THE BASIC EQUATIONS.

Figure (3) indicates the half-space with a system of stationary Cartesian coordinates. The x-axis is in the direction of motion of the step load, the y- and z-axes are normal to the x-axis in and out of the plane of the figure, respectively. The analysis considers the case of plane strain, $\varepsilon_{zz} = 0$, when the velocity V of the step load is larger than the largest possible characteristic or shock velocity which is within the domain of the solution.

The governing equations for the problem are in appearance the same as those for the steady-state two dimensional problem for a conventional elastic material. The equations of motion are

$$\frac{\partial s_x}{\partial x} + \frac{1}{3}\frac{\partial J_1}{\partial x} + \frac{\partial \tau}{\partial y} = \rho \frac{\partial v_x}{\partial t} \tag{7}$$

$$\frac{\partial s_y}{\partial y} + \frac{1}{3}\frac{\partial J_1}{\partial y} + \frac{\partial \tau}{\partial x} = \rho \frac{\partial v_y}{\partial t} \tag{8}$$

The constitutive equations are

$$\dot{s}_x + \frac{2G}{9K}\dot{J}_1 = 2G\frac{\partial v_x}{\partial x} \tag{9}$$

$$\dot{s}_y + \frac{2G}{9K}\dot{J}_1 = 2G\frac{\partial v_y}{\partial y} \tag{10}$$

$$\dot{\tau} = G\left(\frac{\partial v_x}{\partial y} + \frac{\partial v_y}{\partial x}\right) \tag{11}$$

$$\dot{s}_x + \dot{s}_y - \frac{2G}{9K}\dot{J}_1 = 0 \tag{12}$$

In the present problem the moduli G and K are functions of the invariants as described in the Introduction.

Previous experience with steady-state two dimensional problems in inelastic materials, Refs. [4], [5], suggests a procedure for the solution. In brief, the following steps are required:

1. Introduction of the fact that in a steady-state problem all unknowns can only be functions of the variable $\xi = x - Vt$.
2. Elimination of the velocities v_x and v_y .
3. Dimensional considerations, Ref. [4], suggest the possible existence of solutions in which the stresses are solely functions of the variable $\phi = \cot^{-1}(\xi/y)$. Its introduction reduces the partial differential equations to a set of ordinary ones.
4. Finally, the unknowns s_x , s_y and τ are replaced by the principal stress deviators s_1 , s_2 and the angle θ between the direction of s_1 and the horizontal, Fig. (3),

$$s_x = s_1 \cos^2\theta + s_2 \sin^2\theta \tag{13}$$

$$s_y = s_1 \sin^2\theta + s_2 \cos^2\theta \tag{14}$$

$$\tau = (s_1 - s_2) \sin\theta \cos\theta \tag{15}$$

This leads to the four differential equations

$$\begin{bmatrix} -1 & -1 & \frac{2G}{3K} & 0 \\ \sin^2\psi & \cos^2\psi & 1-2X\frac{G_o}{K} & -\sin 2\psi \\ \sin 2\psi & \sin 2\psi & 2\sin 2\psi & -2(1-2X\frac{G_o}{G}) \\ \sin^2\psi - X\frac{G_o}{G} & X\frac{G_o}{G} - \cos^2\psi & -\cos 2\psi & 0 \end{bmatrix} \begin{bmatrix} s_1' \\ s_2' \\ \frac{1}{3}J_1' \\ (s_1 - s_2)\theta' \end{bmatrix} = 0 \qquad (16)$$

where $s_1' \equiv \frac{ds_1}{d\phi}$, etc., and

$$X = \frac{1}{2}\left(\frac{\rho V^2}{G_o}\right)\sin^2\phi \qquad (17)$$

$$\psi = \phi - \theta \qquad (18)$$

Equations (16) are linear and homogeneous, so that the derivatives of the unknowns s_1' , s_2' and J_1' , and the value $(s_1 - s_2)\theta'$ vanish unless the coefficient matrix is singular, requiring

$$\left(2X - \frac{G}{G_o}\right)\left(2X - \frac{K + \frac{4}{3}G}{G_o}\right) = 0 \qquad (19)$$

In any region of finite extent in ϕ, changes in stress or direction θ can thus only occur if one of the two factors of this equation vanishes. Substitution of th respective value of X into Eqs. (16) defines ratios between the rates of change of the stresses and that of θ'. The values of X are

$$2X = \frac{K + \frac{4}{3} G}{G_o} \tag{20}$$

or

$$2X = \frac{G}{G_o} \tag{21}$$

When the last equation applies, the invariant J_1 and thus the value of the mean pressure do not change at all.

Regions in ϕ where changes of stress occur can exist only if $K + \frac{4}{3} G$ and G are, respectively, functions of J_1 and/or J_2 . If one or two of these quantities do not depend on the stresses, the above equations give, instead, one or two values X which, through Eq. (17) define locations ϕ of shock fronts. Equation (20) defines the location of a discontinuity in the normal stress $s_1 + \frac{1}{3} J_1$, while Eq. (21) defines a location of a discontinuity in the shear stress in the plane of the shock front. Both types of shock fronts propagate with their respective characteristic velocity.

If $K + \frac{4}{3} G$ and G are functions of the stresses, discontinuities propagating with noncharacteristic velocities may also occur. The velocity and corresponding locations ϕ of such shocks are not given by Eqs. (20,21), nor defined by the differential equations, Eqs. (16), which apply only in a region of a continuous stress field. The appropriate equations must be obtained for the nonlinear stress-strain relations in combination with momentum considerations (Rankine-Hugoniot relations). The situation is discussed in Appendix A.

III BILINEAR MATERIAL.

In order to obtain insight into the nature of the stress field caused by a progressing step load, Fig. (3), the special case in which the functions k and g in Eqs. (3,4) vanish is considered first. In this case the expressions for K and G do not depend upon the stresses but on whether or not the material is being loaded, unloaded or reloaded. Due to the linear nature of the stress-strain relation, Figs. (1b,2c), this case will be referred to as "bilinear", and the moduli for loading will be designated by K_o and G_o while those for unloading-reloading are $\bar{K}_o$ and $\bar{G}_o$.

As mentioned in Section II, Eqs. (20,2') when applied to the bilinear material do not permit regions of finite extent in ϕ within which the stresses change. Instead, these relations define locations of discontinuities in stress and velocity, similar to dilatational (P) and shear (S) fronts in a conventional linear elastic material. The locations of possible loading fronts are

$$\phi_P = \pi - \sin^{-1} \left[\frac{1}{V} \sqrt{\frac{3K_o + 4G_o}{3\rho}}\right] \tag{22}$$

$$\phi_S = \pi - \sin^{-1} \left[\frac{1}{V} \sqrt{\frac{G_o}{\rho}}\right] \tag{23}$$

In the present problem the only discontinuity for unloading-reloading is an $\bar{S}$ front at

$$\bar{\phi}_S = \pi - \sin^{-1} \left[\frac{1}{V} \sqrt{\frac{\bar{G}_o}{\rho}}\right] \tag{24}$$

As stated previously, only superseismic velocities V will be considered, i.e., the value of V will be sufficiently large so that the angles ϕ_P , ϕ_S , $\bar{\phi}_S$ defined by Eqs. (22,23,24) are real and in the range between $\pi/2$ and π. In addition, due to $\bar{G}_o > G_o$, the inequality $\bar{\phi}_S < \phi_S$ must hold so that the relative locations of the P, S and $\bar{S}$ fronts are shown in Fig. (4).

Stress Changes at $\phi = \phi_P$

The first stress change must be loading because the material is originally unstressed and unstrained. Since $\phi_P < \phi_S$, the first stress change occurs discontinuously at $\phi = \phi_P$. The relation between the principal stresses σ_1 (normal to the front) and σ_2 (parallel to the front) behind the P front, i.e., for $\phi \geq \phi_P$, is

$$\sigma_2 = \frac{3K_o - 2G_o}{3K_o + 4G_o} \sigma_1 \tag{25}$$

and the direction of the principal stress σ_1 is

$$\theta = \phi_P - \frac{\pi}{2} \tag{26}$$

At this stage the value σ_1 is not known and conditions for its determination from the prescribed surface load p must be developed.

Stress Changes at $\phi = \bar{\phi}_S$

The values of the stresses and of the direction θ for $\phi > \phi_P$ must remain as given by Eqs. (25,26) until $\phi = \bar{\phi}_S$

where a second front of discontinuity may occur. The stresses must change at this $\bar{S}$ front in such a way that the material is unloaded and possibly subsequently reloaded as indicated by the value of J_2. To describe the $\bar{S}$ front, the state of stress σ_1, σ_2, θ ahead of $\phi = \bar{\phi}_S$ is resolved into components parallel and perpendicular to $\phi = \bar{\phi}_S$ and the resulting components of stress are denoted by $\bar{\sigma}_T$, $\bar{\sigma}_N$ and $\bar{\tau}_1$, Fig. (5). Since the $\bar{S}$ front involves only shear changes, $\bar{\sigma}_N$ and $\bar{\sigma}_T$ are constant across $\phi = \bar{\phi}_S$ and only the value of $\bar{\tau}_1$ can change to another value $\bar{\tau}_2$. However, since the $\bar{S}$ front is one of unloading-reloading the value of J_2 for $\bar{\tau}_2$ behind the front must not exceed the value of J_2 for $\bar{\tau}_1$ ahead. This implies

$$-\bar{\tau}_1 \leq \bar{\tau}_2 < \bar{\tau}_1 \tag{27}$$

Equation (27) has a geometric interpretation in terms of the principal directions $\bar{\theta}_1$, $\bar{\theta}_2$ on either side of the front. Let δ denote the smaller of the two angles made by the normal to the $\bar{S}$ front and the principal directions ahead of the front. (The two possible cases are shown in Figs. (6a) and (6b).) Equation (27) implies that the appropriate principal direction behind the $\bar{S}$ front makes an angle with the normal to the front which is less than or equal to δ, i.e., it lies in the shaded wedge in the respective Fig. (6a) or (6b). If this shaded wedge includes a vertical line, Fig. (7), a solution involving only the P front and $\bar{S}$ front is possible. In this case the value of the ratio $\bar{\tau}_2/\bar{\tau}_1$ can be

chosen so as to make the appropriate principal axis behind the $\bar{S}$ front vertical, insuring that the boundary condition of vanishing shear stress is satisfied. The value of σ_1 can then be chosen to satisfy the boundary condition $\sigma_y = p$.

It is easy to derive a condition indicating when a solution of the type just described is possible. Figure (7) shows the P and S fronts and the shaded areas which are to contain the vertical. If Fig. (6a) applies, the relation $\bar{\phi}_S - \phi_P \equiv \delta \leq \frac{\pi}{4}$ holds, so that the vertical line will be inside the shaded wedge if $\pi - \bar{\phi}_S \leq \delta \equiv \bar{\phi}_S - \phi_P$. This condition may be written in the form

$$\bar{\phi}_S \geq \frac{\pi + \phi_P}{2} \tag{28}$$

If Fig. (6b) applies no additional condition is required because the wedge always contains the vertical line. Noting that in this case $\bar{\phi}_S > \frac{3\pi}{4}$ and $\bar{\phi}_S \geq \phi_P + \frac{\pi}{4}$ one finds that Eq. (28) is automatically satisfied. It is therefore the general condition for the existence of a solution involving only a P front and an $\bar{S}$ front.

If Eq. (28) is not satisfied, it may be possible to construct solutions by including at $\phi = \phi_S$ an additional shear front at which loading occurs. This possibility, to be considered next, can of course occur only if Fig. (6a) applies.

Stress Changes at $\phi = \phi_S$

Immediately ahead of the S front, Fig. (8), the state of stress is identical to that immediately behind the $\bar{S}$ front. Since the S front inherently involves additional loading, the value of J_2 in the region between the S and $\bar{S}$ fronts must be equal to the previous maximum value of J_2 between the P and $\bar{S}$ fronts. This implies that $\bar{\tau}_2 = -\bar{\tau}_1$ at the $\bar{S}$ front as well as $\tau_2 > \tau_1$ at the S front, where τ_1 and τ_2 are the components of the shear stress parallel to the S front for $\bar{\phi}_S < \phi < \phi_S$ and $\phi > \phi_S$, respectively, Fig. (8). Noting the definition $\delta = \bar{\phi}_S - \phi_P$, the first requirement, $\bar{\tau}_2 = -\bar{\tau}_1$, implies that the direction $\bar{\theta}_2$ of the principal axis behind the $\bar{S}$ front is

$$\bar{\theta}_2 = \bar{\phi}_S + \delta - \frac{\pi}{2} \equiv 2\bar{\phi}_S - \phi_P - \frac{\pi}{2} \tag{29}$$

This is equivalent to saying that the principal axis coincides with the right edge of the shaded wedge in Fig. (6a). The second requirement, $\tau_2 > \tau_1$, implies, Fig. (8), that the angle between the normal to the front and the direction of the principal stresses ahead of and behind the front, $\bar{\theta}_2 - (\phi_S - \frac{\pi}{2})$ and $(\pi - \phi_S)$, respectively, must satisfy the inequality $\bar{\theta}_2 - (\phi_S - \frac{\pi}{2}) < \pi - \phi_S$. In addition, the direction of the principal stress between the S and $\bar{S}$ fronts must be at least as steep as the normal to the S front, or $\bar{\theta}_2 \geq \phi_S - \frac{\pi}{2}$. Substitution of Eq. (29) into the last two inequalities gives

$$\frac{\phi_S + \phi_P}{2} \leq \bar{\phi}_S < \frac{\pi + \phi_P}{2} \tag{30}$$

When this condition is satisfied a solution involving an S front and an $\bar{S}$ front can be constructed.

Summary and Example

Equations (28) and (30) indicate that a solution of one of the two types considered can be constructed whenever $\frac{\phi_S + \phi_P}{2} \leq \bar{\phi}_S$. For example, consider the limits for which the solutions found apply when the values $K_o = 8/3\ G_o$ and $V = 4\sqrt{G_o/\rho}$ are used. Equations (22,23) give $\phi_P = 150.00°$ and $\phi_S = 165.52°$. Therefore, according to Eq. (28), a solution can be constructed involving only a P front and an $\bar{S}$ front if

$$165.00° \leq \bar{\phi}_S < 165.52° \tag{31}$$

while an additional S front is required when, Eq. (30),

$$157.76° \leq \bar{\phi}_S < 165.00° \tag{32}$$

Finally, if $\bar{\phi}_S < 157.76°$ neither of the proposed solutions applies.

Equations (31,32) may be restated in terms of $\bar{G}_o$ by using Eq. (24). It is easily deduced that relation, Eq. (31), holds if $G_o < \bar{G}_o < 1.07\ G_o$, while Eq. (32) holds for $1.07\ G_o < \bar{G}_o < 2.29\ G_o$. For $\bar{G}_o > 2.29\ G_o$ no solution of the type considered exists.

IV NONLINEAR MATERIAL.

A. Material Description

The form of the moduli K and G as functions of the invariants was taken from Ref. [2]. For initial loading,

$$G = G_o + G_1 \left|\sqrt{J_2}\right| - G_2 J_1 + G_3 J_1^2 \qquad (33)$$

$$K = K_o - K_1 \varepsilon_{kk} + K_2 \varepsilon_{kk}^2 \qquad (34)$$

and for unloading and reloading,

$$G = \bar{G}_o + \bar{G}_1 \left|\sqrt{J_2}\right| - \bar{G}_2 J_1 + \bar{G}_3 J_1^2 \qquad (35)$$

$$K = \bar{K}_o - \bar{K}_1 J_1 \qquad (36)$$

The magnitudes and signs of the constants G_j are to be selected such that the stress-strain diagrams have the curvatures and general character shown in Figs. (1a) and (2a).

B. Method of Solution

The configurations of the solutions shown in Figs. (7,8) for the bilinear material are used as guides in the determination of solutions in the nonlinear case. Beginning with a discontinuity in the principal stress normal to the front, $\phi = \phi_p$, Eq. (A-1), it can be expected that instead of the discontinuities in shear in Figs. (7) or (8), the nonlinear material will produce regions in which shear stresses

change in a continuous manner. Because of the change in curvature in the unloading-reloading portion of the diagram, Fig. (2b), there is also a possibility of a noncharacteristic shock in shear adjoining the continuous region.

Conditions which define the discontinuity in normal stress at $\phi = \phi_p$ are discussed in Appendix A. Using an inverse approach, an arbitrary jump σ_1^P in the principal stress σ_1 normal to the front is selected. Integration of the incremental relation at the front, Eq. (A-3) yields the corresponding values σ_2^P and ε_1^P at the front, and hence the starting values

$$\left.\begin{aligned} s_1 &= \frac{2}{3}(\sigma_1^P - \sigma_2^P) \\ s_2 &= -\frac{1}{2}s_1^P \\ J_1 &= \sigma_1^P + 2\sigma_2^P \\ \theta &= \phi_P - \frac{\pi}{2} \\ \varepsilon_1 &= \varepsilon_1^P \end{aligned}\right\} \qquad (37)$$

For locations $\phi > \phi_P$, Eqs. (16)-(19) govern the continuous portion of the solution, where the moduli are given by Eqs. (33)-(36). Equations (16) may be satisfied by regions of uniform stress, $s_1' = s_2' = J_1' = \theta' = 0$, or by regions of stress variation when Eq. (19) is satisfied. Because Eq. (19) implies that the determinant of Eqs. (16) vanishes, only three of the four Equations (16) are thus independent. However, since Eq. (19) must remain valid

throughout regions of stress change, a fourth independent equation in s_1' , s_2' , J_1' and $(s_1 - s_2)\theta'$ is obtained by differentiation of Eq. (19) with respect to ϕ. In the present problem the resulting relation is of interest only for the case of changes in the deviatoric stresses, when $J_1' = 0$. For this case the fourth equation becomes

$$(2s_1 + s_2)s_1' + (2s_2 + s_1)s_2' = \frac{2\rho V^2}{G_1} \left|\sqrt{J_2}\right| \sin 2\phi \qquad (38)^{*)}$$

Substitution of Eqs. (21) and (38) into Eqs. (16) yields the following expressions for the four unknown derivatives

$$s_1' = \frac{2\rho V^2 \left|\sqrt{J_2}\right|}{G_1(s_1 - s_2)} \sin 2\phi \qquad (39)^{*)}$$

$$s_2' = -s_1' \qquad (40)$$

$$J_1' = 0 \qquad (41)$$

$$\theta' = \frac{-s_1'}{(s_1 - s_2) \tan 2\psi} \qquad (42)$$

Equations (39)-(42) permit the numerical determination of the values of stresses in regions of continuous loading or unloading-reloading by quadratures, provided the values on one boundary of the region are known.

The possibility of a noncharacteristic shock in shear associated with the solution must also be explored. From the case of uniaxial pressure waves it is well known that in a material which continuously hardens as the stress level

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*) The value G_1 shown applies for the case of loading. For unloading or reloading it is to be replaced by $\bar{G}_1$.

is changed from level "A" to a level "B", discontinuities with a jump in stress from "A" to "B" can occur. The stress-strain diagram for shear, Fig. (9), indicates that this is the situation in the reloading portion, between points 3 and 4 of the diagram. The complete unloading-reloading diagram has a point of contraflexure at 3. In such a case it is also known from prior experience that there will be a continuous stress change, immediately followed by a discontinuity.

The discontinuity and its velocity can be represented graphically using the stress-strain diagram, Fig. (9). Stress and strain may change discontinuously from a value corresponding to a point 2a, between 2 and 3, to a point 3a, between 3 and 4. Stability of the shock requires that a line 3a to 2a is tangent to the τ-γ diagram at point 2a. The slope of the line defines the shock velocity. As the diagram applies only as far as point 4 the stress level cannot exceed the value at point 4.

If the discontinuity present in a solution does not reach a stress level corresponding to point 4 where the value of J_2 is equal to the previous peak, the solution can not have a region of further loading in shear. If the discontinuity is such that a value J_2 , equal to the previous peak, is reached a region of further loading is possible.

C. Determination and Description of the Solution.

Figure (10) shows the configuration of the solution which will be constructed. It is a generalization of the one in

Fig. (8) for the bilinear material in which two discontinuities in shear occur, one in unloading-reloading, one in further loading.

At ϕ_P the invariant J_2 changes discontinuously from $J_2 = 0$ to a value $\bar{J}_2$. The direction of the principal stress σ_1 is defined by the angle $\theta^P = \phi_P - \frac{\pi}{2}$, while the value of σ_2^P corresponding to the chosen value σ_1^P can be computed from the relations given in Appendix A. Behind the front, $\phi > \phi_P$, there is a region of uniform stress which terminates at the root ϕ_1 of Eq. (21) using the value of G for unloading. At this location a region of continuous unloading of deviatoric and shear stresses begins. Forward integration of Eqs. (39)-(42), using the initial values, Eq. (37), determines the stress field numerically, point by point.

The integration is stopped at a trial location $\phi_2 > \phi_1$ which is selected so that the value of θ, defining the direction of the principal stress, satisfies the inequality $\theta < \phi - \frac{\pi}{2}$. (This ensures that a stable shock in shear in the location ϕ_2 is possible.) As next step, the location $\bar{\phi}_S$ of a shock in shear is determined in which the shear stress changes from the value found at ϕ_2 to the value which makes J_2 equal to the previous maximum of J_2 . The point ϕ_2 and the shock described are accepted if, and only if, $\bar{\phi}_S = \phi_2$, otherwise another point ϕ_2 must be selected, and the process repeated.

For $\phi > \phi_2 = \bar{\phi}_S$ there is again a region in which the principal stresses remain constant. This region can be

terminated at a root $\phi = \phi_3$ of Eq. (21) using G for loading, in which location a new region of continuous stress change begins. Forward integration of Eqs. (39)-(42) proceeds until, at $\phi = \phi_4$, the principal stress σ_1 becomes perpendicular to the surface, i.e., $\theta = \frac{\pi}{2}$. From the point $\phi = \phi_4$ to the surface, $\phi = \pi$, the stresses are again uniform. The value of the principal stress σ_1 obtained in the location ϕ_4 is the value of the applied surface pressure which corresponds to the originally assumed value σ_1^P.

There is also the possibility of a solution which does not contain a region of change associated with further loading in stress between ϕ_3 and ϕ_4. This solution corresponds to the one in Fig. (7) in the bilinear case. In this solution a shock can be selected such that the principal stress for $\phi > \phi_2$ is vertical.

D. Numerical Results

The moduli chosen for the first two numerical examples are

$$\left.\begin{aligned} G &= 7.178 - 20.0 \left|\sqrt{J_2}\right| \\ K &= 11.965 - 150.0\ \varepsilon_{kk} \end{aligned}\right\} \tag{43}$$

for loading, and

$$G = 4.0 + 40.0 \left|\sqrt{J_2}\right| \tag{44}$$

for unloading and reloading.

Because the modulus K affects the solution only during loading at $\phi = \phi_p$ no information for unloading is required. It may be seen from Eq. (39) that solutions in regions of continuous stress changes (in shear) are influenced only by coefficients G_1 and $\bar{G}_1$. Without actual loss of generality G_2 , G_3 , $\bar{G}_2$, $\bar{G}_3$ were conveniently assumed to be zero.

Figures (11) and (12) show numerical results for the moduli given above for two different ratios of surface pressure speed, V, to the speed c_p . It may be noted that the regions of continuous stress change are quite small in angular extent. This reflects the relatively small amount of curvature of the stress-strain diagrams for the moduli used. There is a relatively small change in σ_1 from the value at $\phi = \phi_p$ to the surface value at $\phi = \pi$. However, if stress components in the horizontal and vertical directions, σ_x and σ_y respectively, are considered [see Figs. (13) and (14)] the changes in the stress field are more significant and visible.

Figure (15) shows numerical results for the modulus

$$G = 4.0 + 10.0 \left|\sqrt{J_2}\right| \tag{45}$$

for unloading and reloading. The initial loading moduli were chosen to be the same as Eqs. (43). In this case only one region of continuous stress change is present. The stress changes introduced by the shock in shear are sufficient to satisfy the boundary conditions, and no region of further

loading is ~equired. This alternate solution, corresponds to the one shown in Fig. (7) for a bilinear material. It occurs if the parameters are such that a shock in shear can be selected so that the principal stress σ_1 becomes vertical, $\theta = \frac{\pi}{2}$, immediately behind the front. In this case the front is represented by the line 2a-3a in Fig. (9).

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[1] "Investigation of Ground Shock Effects in Nonlinear Hysteretic Media - Report 1 - Development of Mathematical Material Models", by I. Nelson and M.L. Baron, U.S. Army Engineer Waterways Experiment Station, Vicksburg, Mississippi, Contract DACA39-67-C-0048, Report S-68-1, March 1968.

[2] "Investigation of Ground Shock Effects in Nonlinear Hysteretic Media - Report 2 - Modeling the Behavior of a Real Soil", by I. Nelson, U.S. Army Engineer Waterways Experiment Station, Vicksburg, Mississippi, Contract DACA39-67-C-0048, forthcoming report.

[3] "Investigation of Ground Shock Effects in Nonlinear Hysteretic Media - Report 3 - A Note on the Plane Waves of Pressure and Shear in a Half-Space of Hysteretic Material", by A. Matthews and H.H. Bleich, U.S. Army Engineer Waterways Experiment Station, Vicksburg, Mississippi, Contract DACA39-67-C-0048, January 1969.

[4] "Step Load Moving with Superseismic Velocity on the Surface of an Elastic-Plastic Half-Space", by H.H. Bleich and A.T. Matthews, International Journal of Solids and Structures, 1967, Vol. 3, pp 819 to 852.

[5] "Moving Step Load on the Surface of a Half-Space of Granular Material", by H.H. Bleich, A.T. Matthews and J.P. Wright, International Journal of Solids and Structures, 1968, Vol. 4, pp 243 to 286.

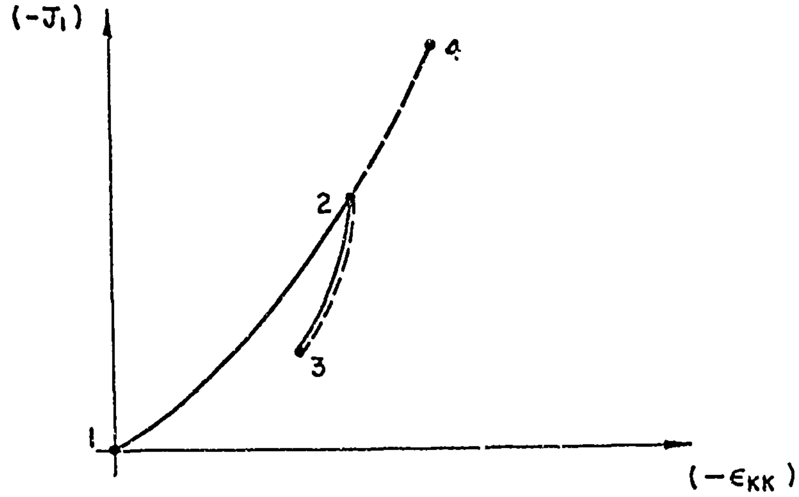


FIG. 1a

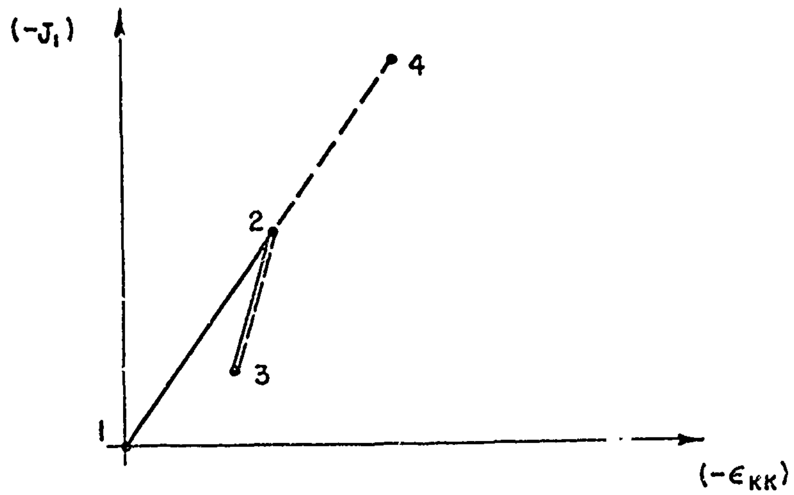


FIG. 1b

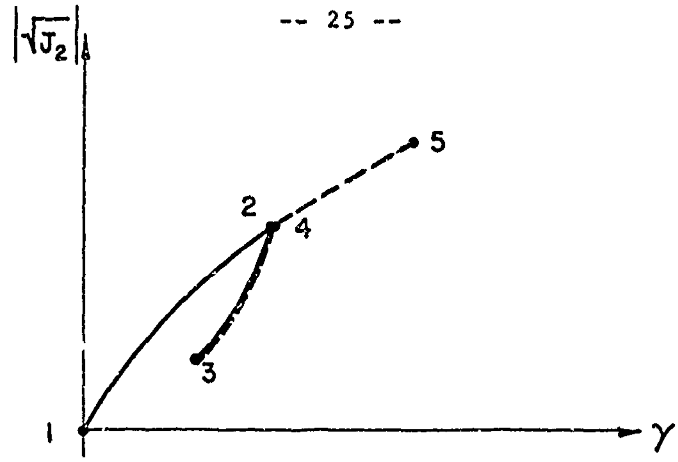


FIG. 2a

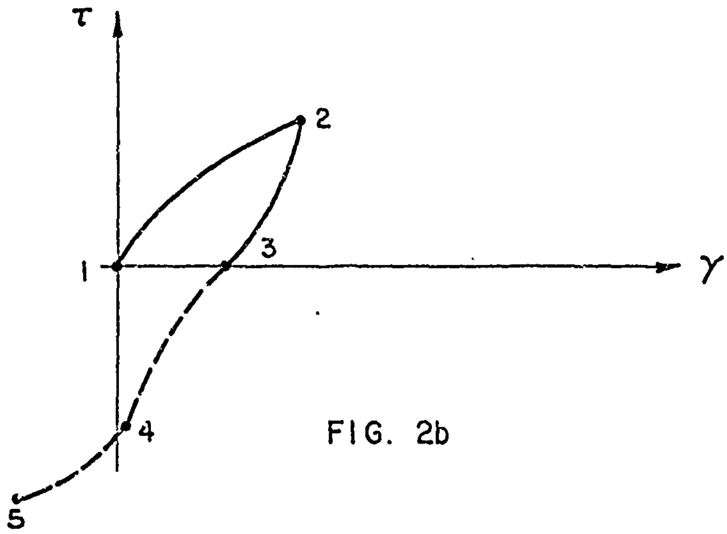


FIG. 2b

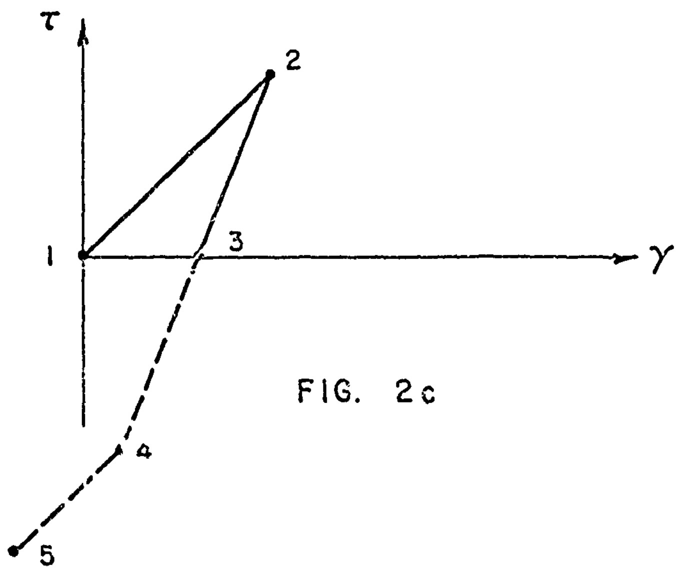


FIG. 2c

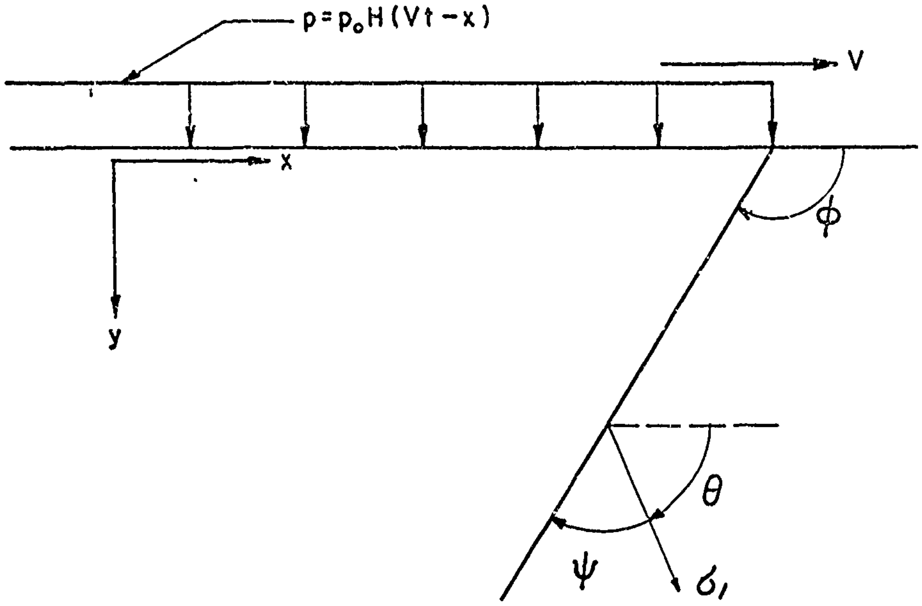


FIG. 3

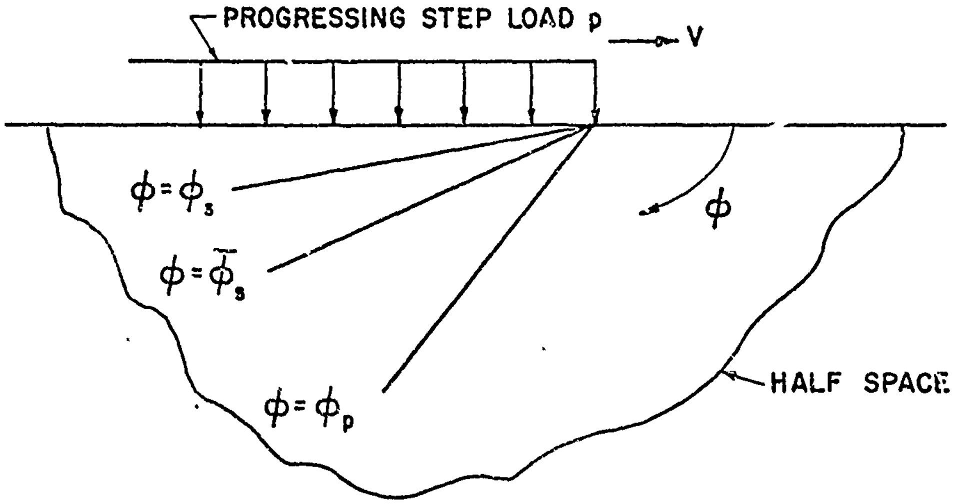
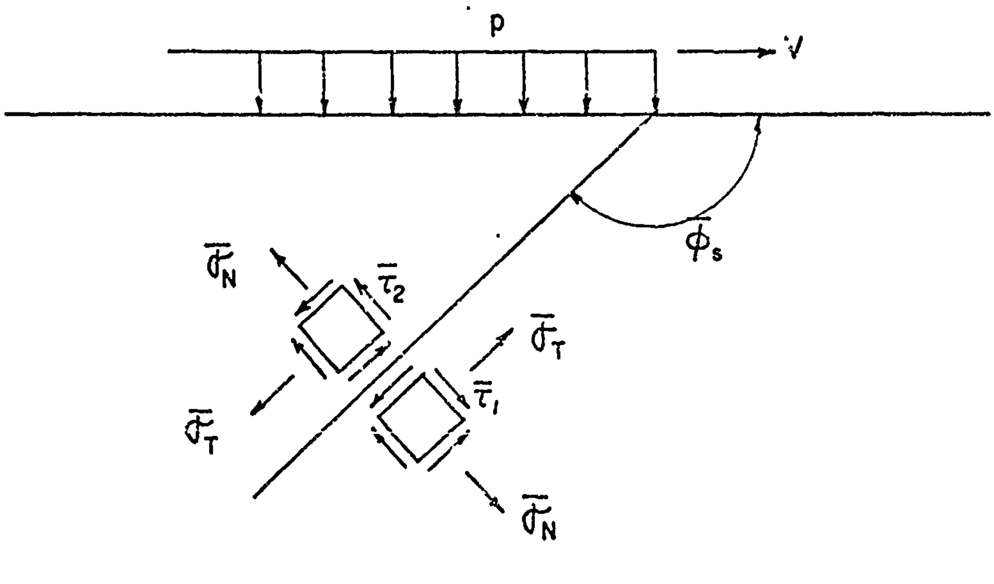


FIG. 4



(DIRECTIONS SHOWN FOR POSITIVE σ, τ)

FIG. 5

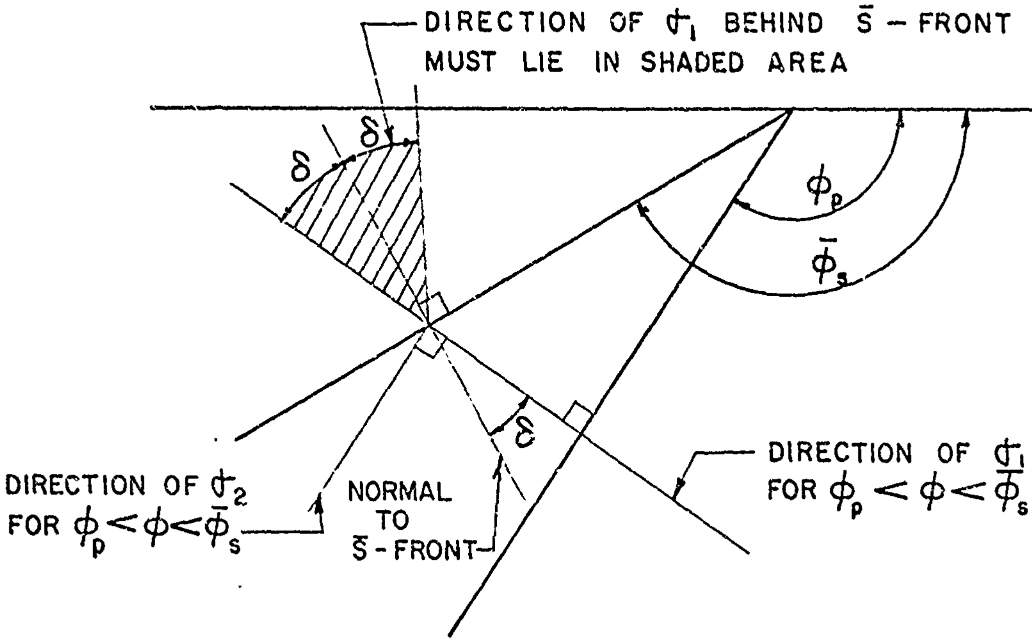

DIRECTION OF σ_1 BEHIND $\bar{S}$ – FRONT MUST LIE IN SHADED AREA
δ
δ
ϕ_p
$\bar{\phi}_s$
δ
DIRECTION OF σ_1 FOR $\phi_p < \phi < \bar{\phi}_s$
DIRECTION OF σ_2 FOR $\phi_p < \phi < \bar{\phi}_s$
NORMAL TO $\bar{S}$ - FRONT

FIG. 6a

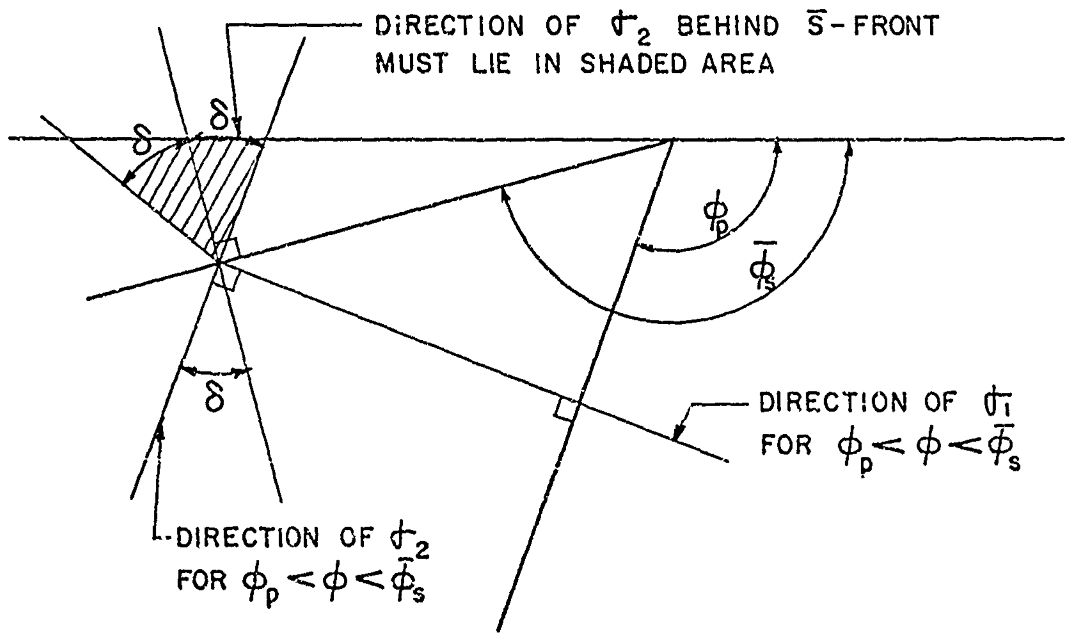

DIRECTION OF σ_2 BEHIND $\bar{S}$ - FRONT MUST LIE IN SHADED AREA
δ
δ
ϕ_p
$\bar{\phi}_s$
δ
DIRECTION OF σ_1 FOR $\phi_p < \phi < \bar{\phi}_s$
DIRECTION OF σ_2 FOR $\phi_p < \phi < \bar{\phi}_s$

FIG. 6b

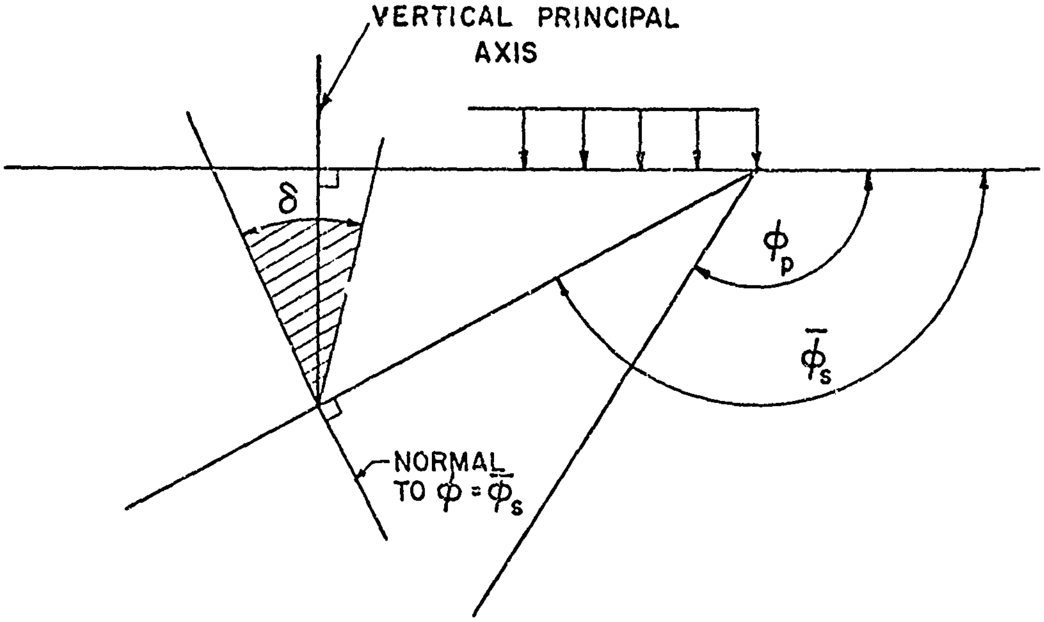
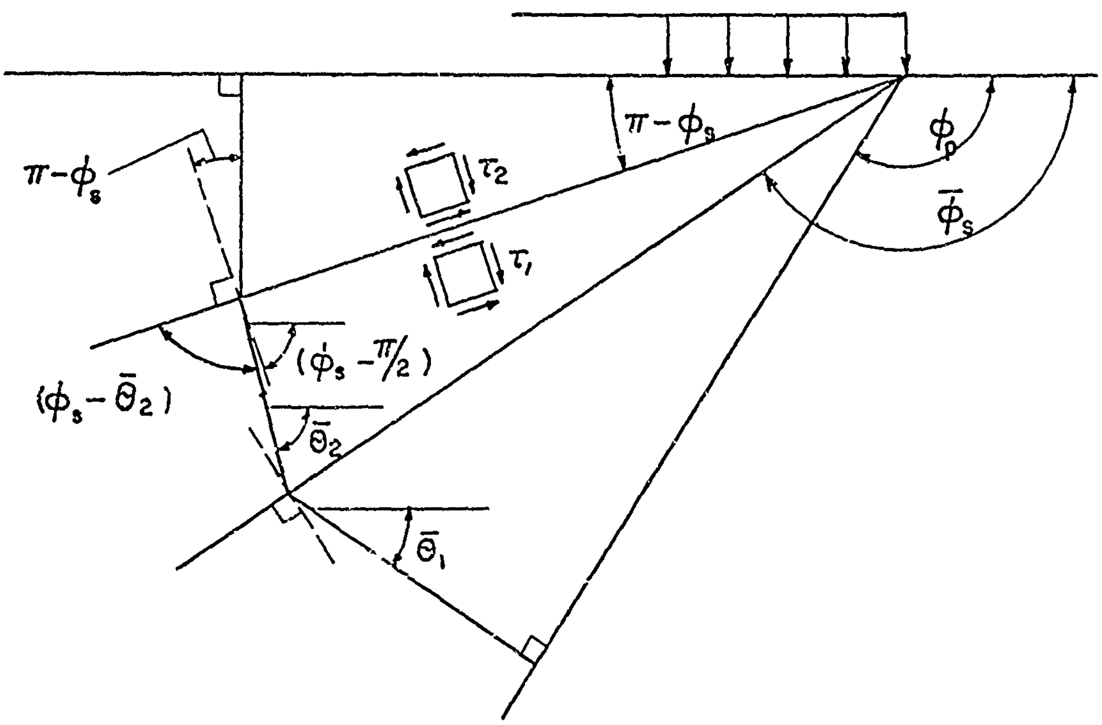


FIG. 7



(DIRECTION SHOWN FOR POSITIVE τ)

FIG. 8

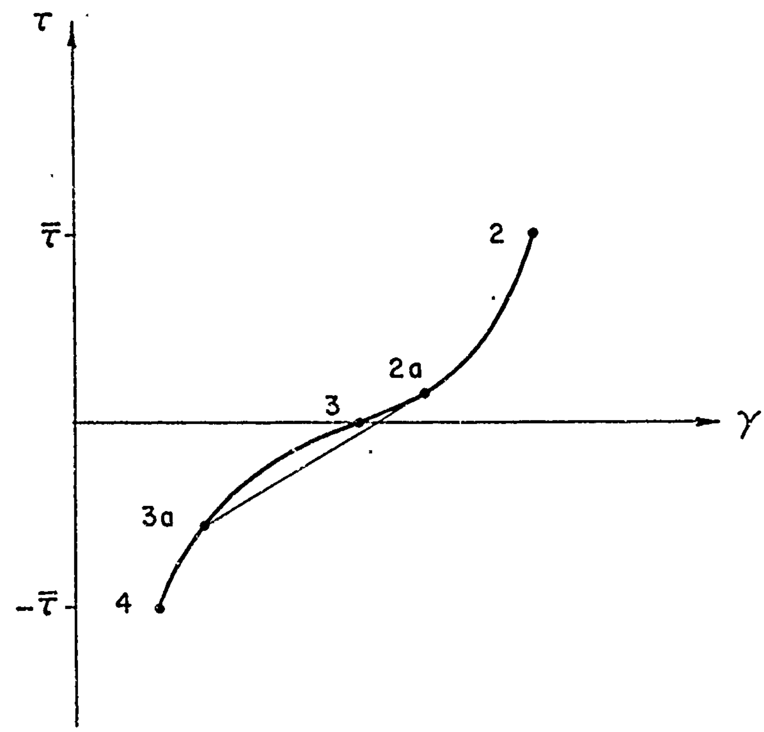


FIG. 9

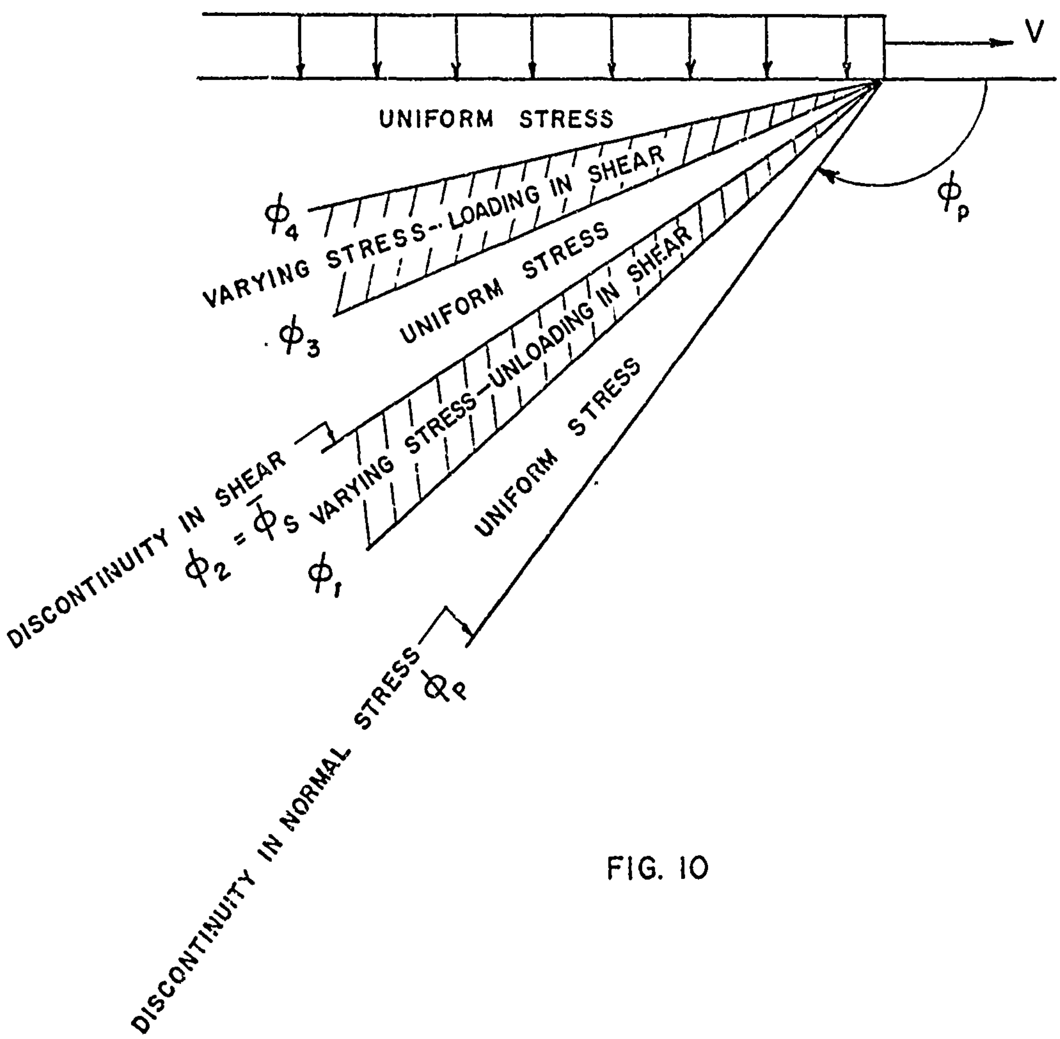

V
UNIFORM STRESS
ϕ_4
VARYING STRESS-- LOADING IN SHEAR
ϕ_p
UNIFORM STRESS
ϕ_3
VARYING STRESS-UNLOADING IN SHEAR
UNIFORM STRESS
DISCONTINUITY IN SHEAR
$\phi_2 = \bar{\phi}_S$
ϕ_1
DISCONTINUITY IN NORMAL STRESS
ϕ_P

FIG. 10

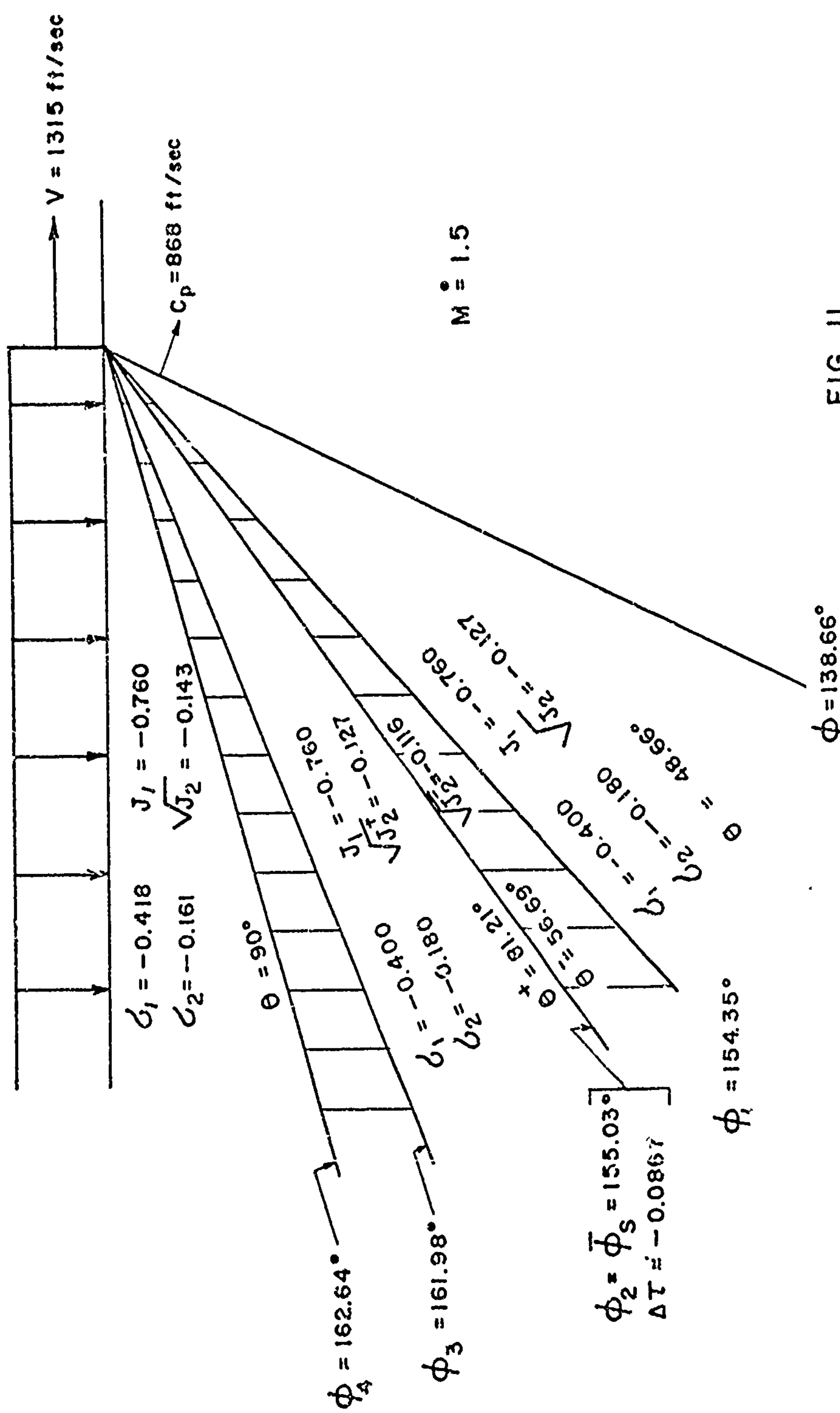

V = 1315 ft/sec
C_p = 868 ft/sec
M = 1.5
σ_1 = -0.418
σ_2 = -0.161
J_1 = -0.760
√J_2 = -0.143
θ = 90°
σ_1 = -0.400
σ_2 = -0.180
J_1 = -0.760
√J_2 = -0.127
√J_2 = -0.116
θ* = 81.21°
θ = 56.69°
J_1 = -0.760
√J_2 = -0.127
σ_1 = -0.400
σ_2 = -0.180
θ = 48.66°
φ_p = 138.66°
φ_1 = 154.35°
φ_2 = Φ_S = 155.03°
ΔT' = -0.0867
φ_3 = 161.98°
φ_4 = 162.64°

FIG. 11

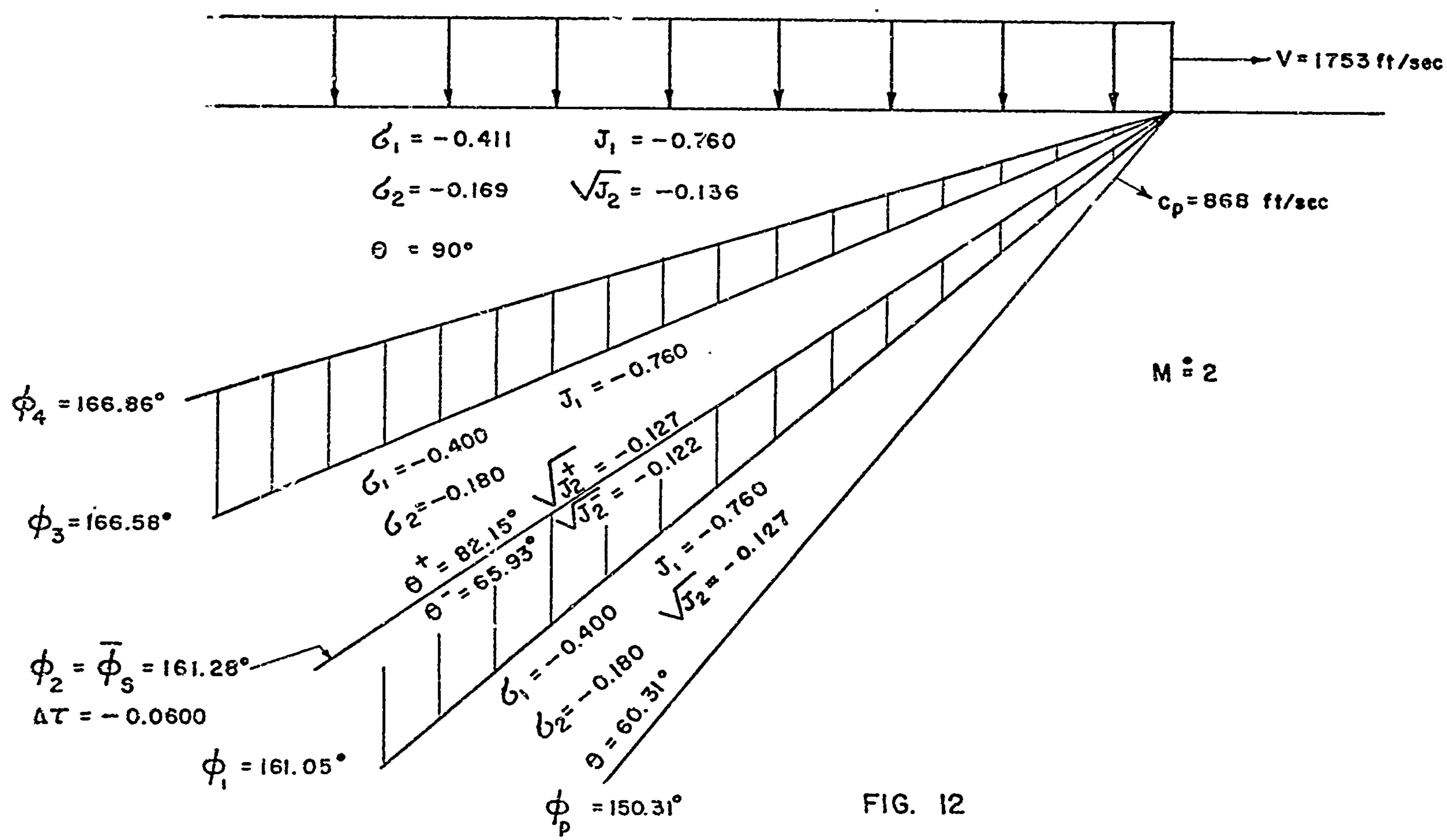

V = 1753 ft/sec
c_p = 868 ft/sec
M = 2
$\sigma_1 = -0.411$
$\sigma_2 = -0.169$
$\theta = 90°$
$J_1 = -0.760$
$\sqrt{J_2} = -0.136$
$J_1 = -0.760$
$\sigma_1 = -0.400$
$\sigma_2 = -0.180$
$\sqrt{J_2^+} = -0.127$
$\sqrt{J_2^-} = -0.122$
$\theta^+ = 82.15°$
$\theta^- = 65.93°$
$J_1 = -0.760$
$\sqrt{J_2} = -0.127$
$\sigma_1 = -0.400$
$\sigma_2 = -0.180$
$\theta = 60.31°$
$\phi_4 = 166.86°$
$\phi_3 = 166.58°$
$\phi_2 = \bar{\phi}_S = 161.28°$
$\Delta\tau = -0.0600$
$\phi_1 = 161.05°$
$\phi_p = 150.31°$

FIG. 12

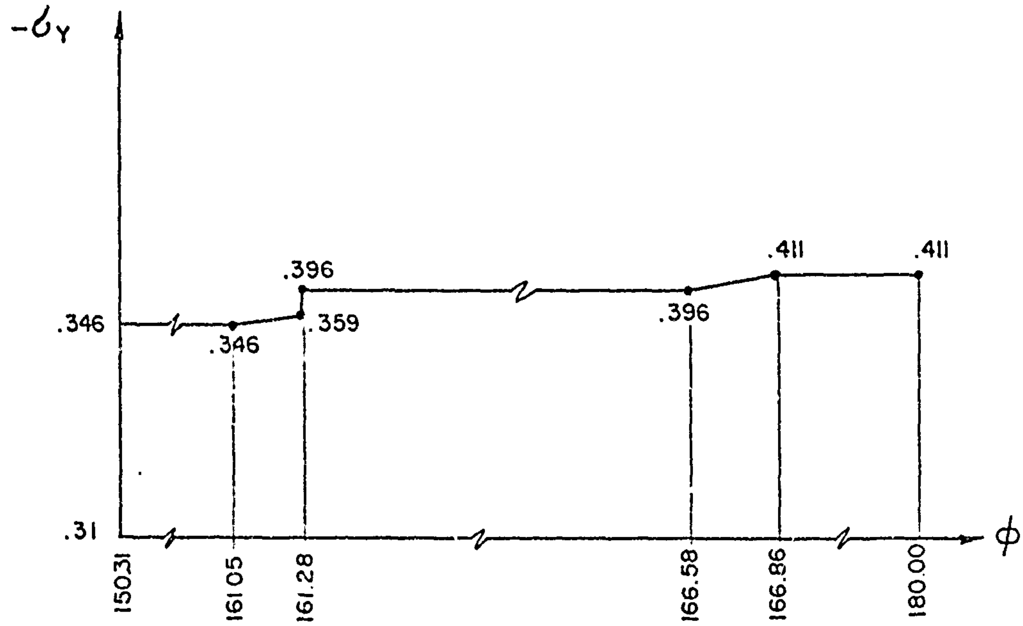

$-\sigma_Y$
.346
.31
.396
.346
.359
.396
.411
.411
ϕ
150.31
161.05
161.28
166.58
166.86
180.00

FIG. 13

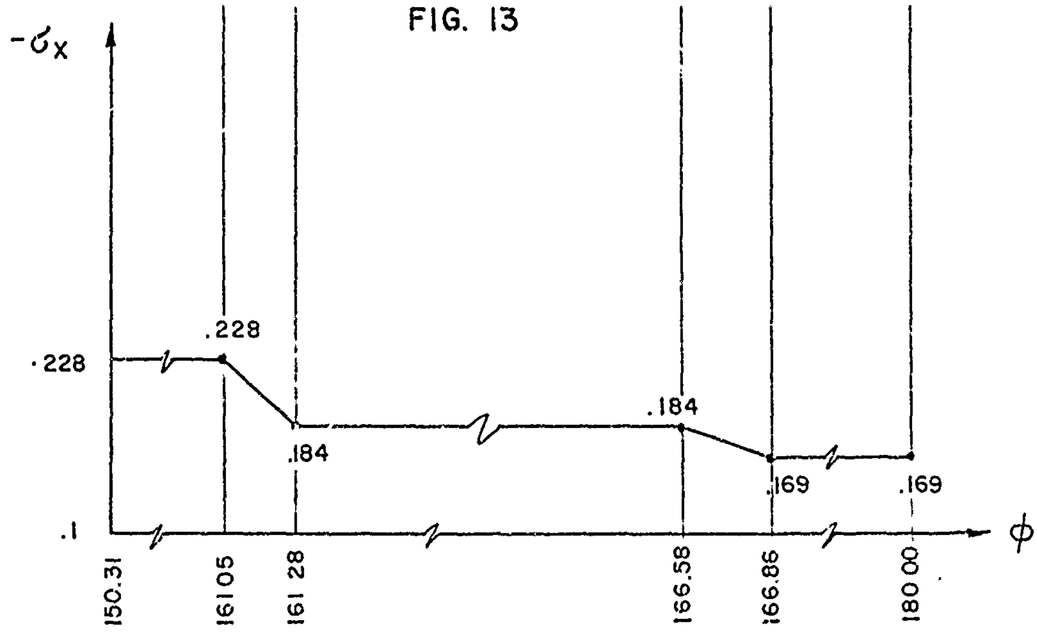

$-\sigma_X$
.228
.1
.228
.184
.184
.169
.169
ϕ
150.31
161.05
161.28
166.58
166.86
180.00

FIG. 14

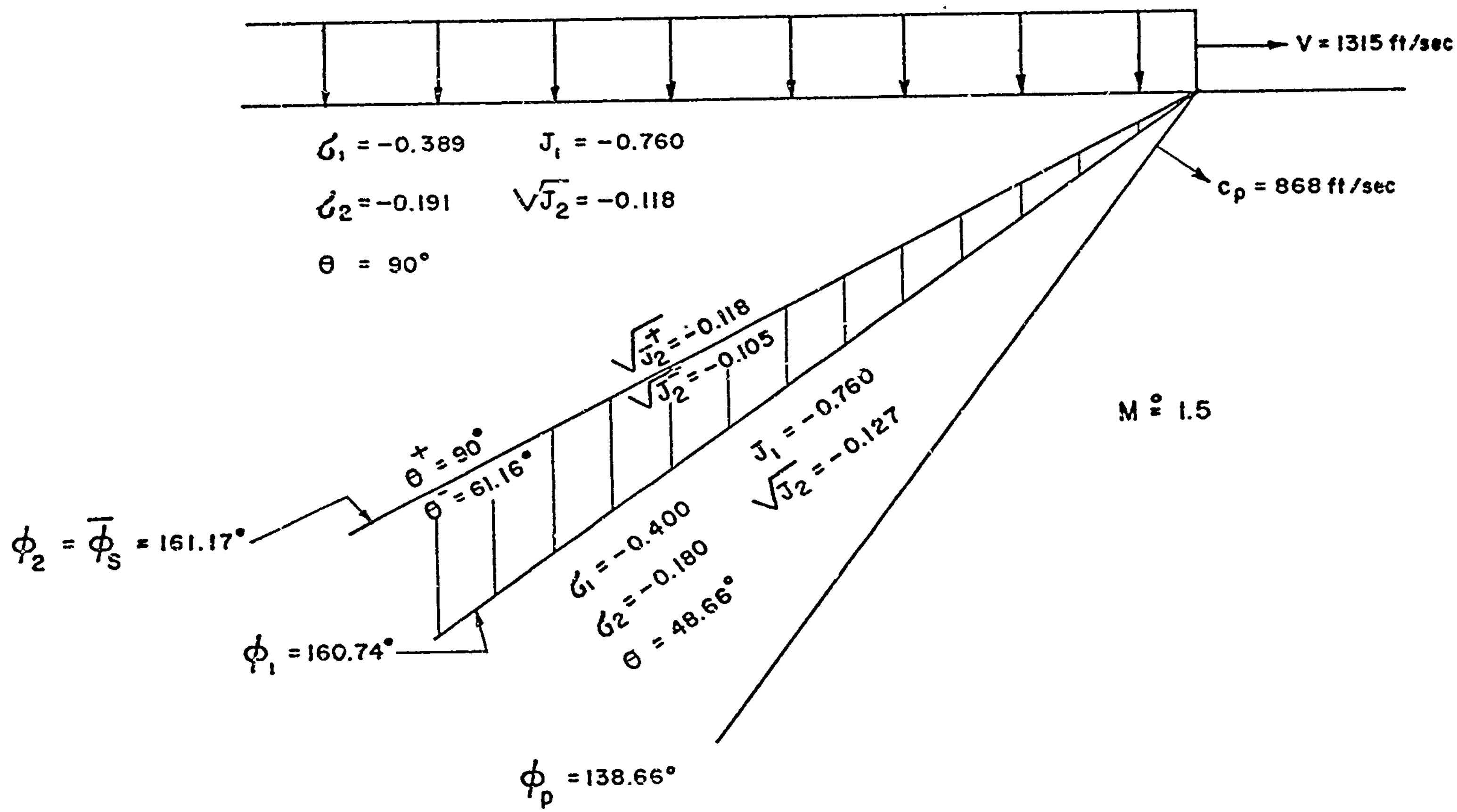

V = 1315 ft/sec
c_p = 868 ft/sec
σ_1 = -0.389
σ_2 = -0.191
θ = 90°
J_1 = -0.760
$\sqrt{J_2}$ = -0.118
$\sqrt{J_2^+}$ = -0.118
$\sqrt{J_2^-}$ = -0.105
θ^+ = 90°
θ^- = 61.16°
J_1 = -0.760
$\sqrt{J_2}$ = -0.127
σ_1 = -0.400
σ_2 = -0.180
θ = 48.66°
M = 1.5
$\phi_2 = \overline{\phi}_s$ = 161.17°
ϕ_1 = 160.74°
ϕ_p = 138.66°

FIG. 15

APPENDIX A

Consider a plane moving front at which discontinuities in stresses and velocities occur, and let ξ,η,ζ be a Cartesian coordinate system, Fig. (A-1), where ξ is normal, and η,ζ tangential to the front. Conditions for discontinuities of stress and velocity normal as well as tangential to the front will be studied, but the two cases must be treated separately.

1. The P-front.

A front where discontinuities of stress and velocity occur in the direction ξ of propagation of the front will be referred to as a P-front. Let $\Delta\sigma_\xi$, Δv_ξ be the discontinuous changes of normal stress and velocity, respectively. Conservation of momentum at the front requires

$$\Delta\sigma_\xi = \rho c_p \Delta v_\xi \tag{A-1}$$

where c_p is the velocity of propagation of the front, while

$$\left.\begin{aligned} \Delta\varepsilon_\xi &= \frac{1}{c_p}\,\Delta v_\xi \\ \Delta v_\eta &= \Delta v_\zeta = 0 \\ \Delta\sigma_\eta &= \Delta\sigma_\zeta \neq 0 \end{aligned}\right\} \tag{A-2}$$

Equation (A-1) and the first of Eqs. (A-2) give the relation $c_p^2 = \dfrac{\Delta\sigma_\xi}{\rho\Delta\varepsilon_\xi}$. The inclination ϕ_p of such a front in the problem considered in the body of the report is thus

$$\phi_p = \pi - \sin^{-1}\left[\frac{1}{V}\sqrt{\frac{\Delta\sigma_\xi}{\rho\Delta\varepsilon_\xi}}\right] \tag{A-3}$$

Consider in detail the case where the region ahead of the front is stressless and at rest. The normal to the front is then a principal direction, i.e., $\sigma_1 = \sigma_\xi$, $\sigma_2 = \sigma_\eta$, $\sigma_3 = \sigma_\zeta$. To find for any selected value $\Delta\sigma_1$ the jumps in the other quantities Eqs. (1) and (2) are combined with Eqs. (A-1) and (A-2) to obtain the "one dimensional" incremental stress-strain relation

$$\dot{\sigma}_1 = (K + \frac{4}{3} G)\ \dot{\varepsilon}_1 \qquad \text{(A-4)}$$

Using the values of K and G given by Eqs. (33) and (34) the integral of $\dot{\sigma}_2$ can be found in closed form. First Eq. (1) is integrated to express J_1 as a function of ε_{kk} . The result is substituted into Eqs. (5), (6) and (33), then into Eq. (A-4), with the result

$$\frac{d\sigma_2}{d\varepsilon_1} + a\sigma_2 = \sum_{j=1}^{7} C_j\ \varepsilon_1^{j-1} \qquad \text{(A-5)}$$

where

$$\left.\begin{aligned}
a &= \frac{2}{3}\sqrt{3}\ G_1 \\
C_1 &= K_o - \frac{2}{3} G_o \\
C_2 &= -K_1 + aK_o + 2G_2K_o \\
C_3 &= K_2 - \frac{1}{2} aK_1 - G_2K_1 - 6G_3K_o^2 \\
C_4 &= \frac{1}{3} aK_2 + \frac{2}{3} G_2K_2 + 6G_3K_oK_1 \\
C_5 &= -4K_oK_2G_3 - \frac{3}{2} G_3K_1^2 \\
C_6 &= 2G_3K_1K_2 \\
C_7 &= -\frac{2}{3} G_3K_2^2
\end{aligned}\right\} \qquad \text{(A-6)}$$

The solution of the differential equation (A-5), for the initial condition $\sigma_2 = 0$ when $\varepsilon_1 = 0$, is

$$\sigma_2 = -2 \sum_{m=1}^{7} \sum_{j=m}^{7} \frac{(-1)^{j-m}(j-1)!}{a^{j-m+1}(m-1)!} C_j(1-\delta_{1m}e^{-a\varepsilon_1})\ \varepsilon_1^{m-1} + 3K_o\varepsilon - $$
$$-1.5\ K_1\varepsilon_1^2 + K_2\varepsilon_1^3 \qquad \text{(A-7)}$$

where $\delta_{1m} = 1$ for $m=1$, $\delta_{1m} = 0$ for $m \neq 1$.

A Newton Raphson technique was used in the examples to find the value of ε_1 for the selected values of σ_2 . σ_1 may then be found from the integral of Eq. (1) giving the complete stress system at the front.

2. The S-front.

Discontinuous changes at the S-front occur only in the velocity v_η parallel to the front, and in the corresponding stress $\sigma_{\xi\eta} = \tau$. Conservation of momentum at the front requires

$$\Delta\tau = \rho c_S \Delta v_\eta \qquad \text{(A-8)}$$

where c_S is the velocity of propagation of the front, while

$$\left.\begin{aligned} \Delta\gamma &= \frac{1}{2c_S}\,\Delta v_\eta \\ \Delta v_\xi &= \Delta v_\zeta = 0 \\ \Delta\sigma_\xi &= \Delta\sigma_\eta = \Delta\sigma_\zeta = 0 \end{aligned}\right\} \qquad \text{(A-9)}$$

Equation (A-8) and the first of Eqs. (A-9) give the relation $c_S^2 = \frac{\Delta\tau}{2\rho\Delta\gamma}$. The inclination ϕ_S of such a front in the problem

considered in the body of the report is thus

$$\phi_S = \pi - \sin^{-1} \left[\frac{1}{V} \sqrt{\frac{\Delta\tau}{2\rho\Delta\gamma}}\right] \qquad \text{(A-10)}$$

To determine the ratio $\Delta\tau/\Delta\gamma$ consider the case where the front propagates into a region which has an existing stress field with principal directions not perpendicular to the front. The incremental stress-strain relation is

$$\dot{\tau} = 2G\,\dot{\gamma} \qquad \text{(A-11)}$$

where G depends on the invariant

$$J_2 = s_\xi^2 + s_\xi s_\eta + s_\eta^2 + \tau^2 \qquad \text{(A-12)}$$

As stated in the body of the report, only discontinuities in shear for unloading - reloading were required. For this situation Eq. (A-11) was integrated numerically using Eq. (35) for G.

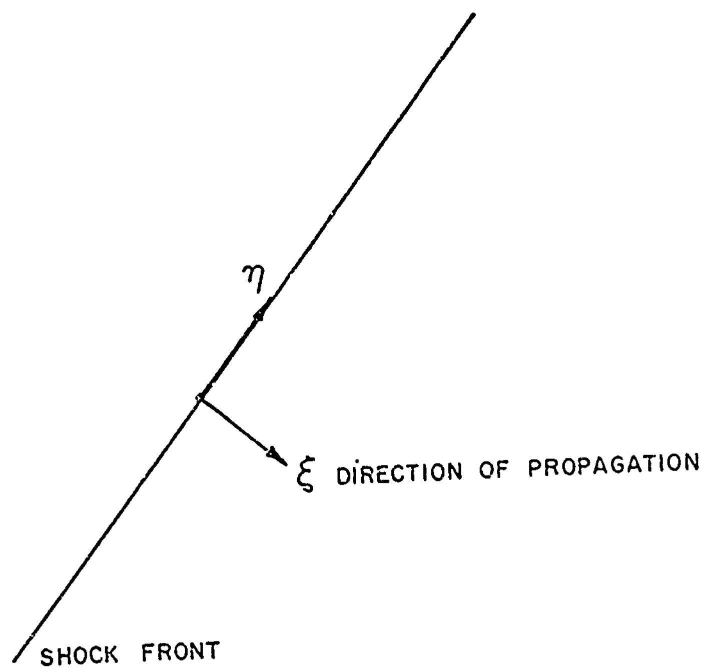


FIG. A-1

APPENDIX B

Bilinear Fluid.

The simplest case of the new material model considered in this report is that of a bilinear fluid with a pressure-volume relation similar to the one shown in Fig. (1b) for the solid. If a half-space of such a material is subjected to a supersonic progressing step load, the solution is trivial. It consists of a shock front (in which the lower modulus K_L for loading applies) follo ed by a uniform stress field. The higher value of the modulus, K, do s not affect the solution at all. The simpler nature of the bilinear fluid, as compared to the solid, makes it possible to give a closed form solution for the alternative problem of a half-space, Fig. (B-1), subject to a progressing pressure pulse which rises suddenly to a value P_o, but subsequently decreases linearly. This problem involves both loading and unloading in the fluid, so that the nature of the material affects the result.

The basic equations describing an inviscid fluid may be written in nondimensional form by introducing the following units of length, velocit, and pressure, respectively,

length: $\frac{P_o}{\lambda}$ = $\frac{\text{discontinuity in surface pressure}}{\text{gradient of surface pressure}}$

velocity: V = velocity of surface load

pressure: $\rho_o V^2$

where ρ_o is the nominal density of the fluid. The linearized nondimensional equations of conservation of mass and momentum

are

$$\left.\begin{aligned} &\frac{\partial p}{\partial x} + \frac{\partial u}{\partial t} = 0 \\ &\frac{\partial p}{\partial y} + \frac{\partial v}{\partial t} = 0 \\ &\frac{\partial p}{\partial t} + \left(\frac{K}{\rho_o V^2}\right)\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right) = 0 \end{aligned}\right\} \qquad \text{(B-1)}$$

where p is the pressure in the fluid while u and v are the horizontal and vertical components of velocity, respectively. In this form the equations depend solely on the nondimensional parameter $\frac{K}{\rho_o V^2}$. This parameter varies because K assumes the value K_L or K_U depending upon whether the fluid is being loaded or unloaded - reloaded, respectively.

In the steady-state problem considered here, p, u and v are necessarily functions of y and the combination $\xi = x-t$. After elimination of u, Eqs. (B-1) can therefore be reduced to

$$\left.\begin{aligned} &\frac{\partial p}{\partial y} - \frac{\partial v}{\partial \xi} = 0 \\ &Z^2 \frac{\partial p}{\partial \xi} - \frac{\partial v}{\partial y} = 0 \end{aligned}\right\} \qquad \text{(B-2)}$$

where Z is the nondimensional parameter,

$$Z = \sqrt{\frac{\rho_o V^2}{K} - 1} \qquad \text{(B-3)}$$

which may assume the alternative values Z_L , Z_U for loading and unloading - reloading, respectively.

Consider the situation when V is supersonic with respect to the sound speeds for loading, $\sqrt{\frac{K_L}{\rho_o}}$, and unloading,

$\sqrt{\frac{K_U}{\rho_o}} > \sqrt{\frac{K_L}{\rho_o}}$. If the fluid were linear with $K = K_L$, instead of bilinear, one finds easily that the solution consists of a shock front inclined at the angle $\phi_P = \pi - \sin^{-1}\left[\frac{1}{V}\sqrt{\frac{K}{\rho_o}}\right]$ followed by gradually (as a matter of fact linearly) decreasing pressure. It will be assumed, and subsequently verified, that the situation for the bilinear fluid is similar, i.e., there is loading at the front only, and unloading everywhere else.

As the entire region between the surface and the P-front, Fig. (B-2), is assumed to be an unloading region, the problem requires the solution to the hyperbolic differential equations (B-2) for $Z = Z_U$ in this region. This is now a linear hyperbolic problem. As the boundary conditions for this problem, one imposes at all points Q on the loading front, i.e., $\xi_Q = -Z_L y_Q$, the momentum requirement

$$v_Q = Z_L p_Q \tag{B-4}$$

while at any surface point T with the coordinate ξ_T , the surface pressure is specified,

$$p_T = p_o(1 + \xi_T) \tag{B-5}$$

The linear form of the boundary conditions, Eqs. (B-4) and (B-5), in this linear hyperbolic problem implies that the pressure $p(\xi,y)$ is of the linear form

$$p(\xi,y) = p_o(1 + \xi + ky) \tag{B-6}$$

where the coefficient of ξ is unity in order to satisfy

Eq. (B-5), while the constant k is so far undetermined. Substitution of Eq. (B-6) into Eqs. (B-2) leads to

$$v(\xi,y) = p_o(v_o + k\xi + Z_U^2 y) \tag{B-7}$$

where v_o is a constant of integration. The values of k and v_o are determined by substitution of Eqs. (B-6) and (B-7) into Eq. (B-4),

$$\left.\begin{aligned} k &= \frac{Z_U^2 + Z_L^2}{2Z_L} \\ v_o &= Z_L \end{aligned}\right\} \tag{B-8}$$

In order to verify the premise that the entire wedge between the P-front and the surface is continuously unloading consider the pressure, obtained from Eq. (B-6), $p = p_o(1 + x - t + ky)$. Since $\frac{\partial p}{\partial t}$ is obviously negative, the validity of the solution is confirmed. Equations (B-6), (B-7), (B-8) and Fig. (B-2) thus describe the solution to the problem.

The solution presented above, while intended for completely supersonic velocities V of the surface pressure, appears to remain valid when the load moves subsonically with respect to the higher of the sound speeds, $\sqrt{\frac{K_U}{\rho_o}}$, provided that V is still supersonic with respect to the sound speed in loading, $\sqrt{\frac{K_L}{\rho_o}}$. It is of general interest to show that in this case, where the situation is not completely supersonic, the above solution, while valid, is not unique. Alternate solutions are obtained by adding terms which satisfy homogeneous boundary conditions on the surface and on the P-front.

For the case now being considered, Eq. (B-3) gives a negative value for Z_U^2 while Z_L^2 remains positive. Introduce the real and positive quantity Y,

$$Y^2 = -Z_U^2 \quad ; \quad Y > 0 \tag{B-9}$$

and a new set of independent variables,

$$\left.\begin{aligned} r &= \sqrt{\frac{\xi^2}{Y^2} + y^2} \\ \theta &= \tan^{-1}\left(-\frac{yY}{\xi}\right) \end{aligned}\right\} \tag{B-10}$$

It may be verified easily that

$$\left.\begin{aligned} p &= \frac{1}{Y}\frac{\partial\phi}{\partial y} \\ v &= -Y\frac{\partial\phi}{\partial\xi} \end{aligned}\right\} \tag{B-11}$$

with

$$\phi = r^\lambda \cos \lambda\theta \tag{B-12}$$

is a solution of Eqs. (B-2) which satisfies the homogeneous boundary condition p = 0 on the surface θ = 0 for all λ. The constant λ can be used to satisfy the remaining conditions, Eq. (B-4). If singularities in pressure and velocity are eliminated as inappropriate, one obtains an infinite number of roots $\lambda = \lambda_n$, namely,

$$\lambda_n = \frac{n\pi}{\tan^{-1}\left(\frac{Y}{Z_L}\right)} \quad ; \quad n = 1, 2, \ldots \tag{B-13}$$

where the principal value of $\tan^{-1}$ is used. Therefore, one

may add to the previous solution, Eqs. (B-6), (B-7), (B-8) and (B-9), a term of the form of Eqs. (B-11) where

$$\phi = \sum_{n=1}^{\infty} A_n r^{(\lambda_n)} \cos \lambda_n \theta \qquad \text{(B-14)}$$

The coefficients A_n are arbitrary, except for the restriction that $\frac{\partial p}{\partial t} < 0$ everywhere. Steady-state solutions are thus not unique when the surface load moves subsonically with respect to the speed of sound which applies for unloading - reloading.

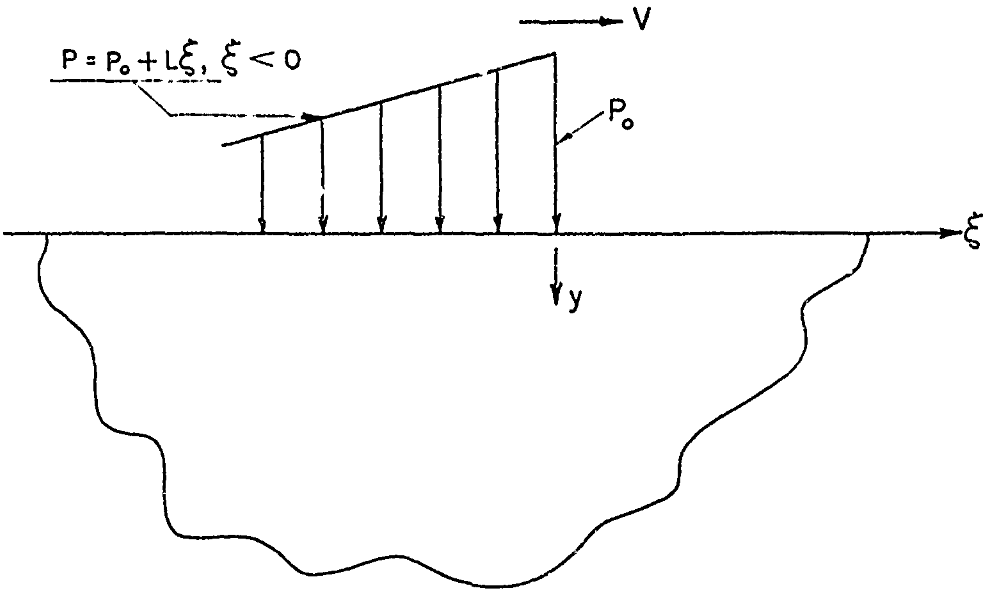
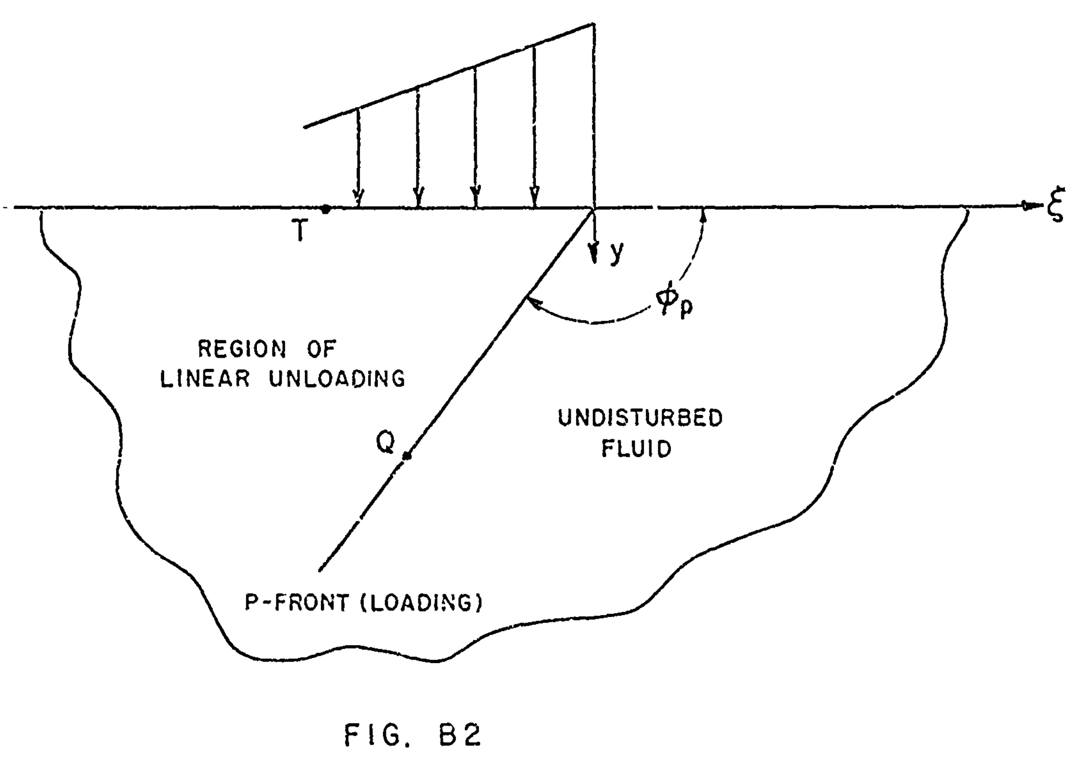

V
$P = P_o + L\xi,\ \xi < 0$
P_o
ξ
y

FIG. B1


ξ
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REGION OF
LINEAR UNLOADING
UNDISTURBED
FLUID
Q
P-FRONT (LOADING)

FIG. B2

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13. ABSTRACT

The report considers the two dimensional effects of a step wave progressing with constant superseismic velocity on the surface of a half-space. The material treated is isotropic and it is assumed that incremental relations between both deviatoric and volumetric stress and strain depend not only on the instantaneous values of those quantities, but also on bulk and shear moduli which differ according to whether initial loading, unloading, or reloading occurs.

Simple closed form solutions are obtained when the moduli K and G are constant and lie within certain limits. For the more general case, when K and G are functions of the first and second invariants of stress, solutions requiring only quadratures are found.

As an auxiliary study, a problem involving a half-space of fluid with a bilinear pressure volume relation is solved.

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